Electromagnetic Design of flexIble SensOrs

Report 7. Hybrid method (mode matching & FEM)

BSc. Malgorzata Warecka

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1 Introduction

The aim of the report is to present hybrid method that combine finite element method and mode matching method. In this report to solve the problem the Z matrix is used. Using this approach the far field can be obtained as well as the near field and no open space boundary conditions are needed. The scattering problem is defined in two dimensions. The structure is homogeneous in the direction of the z axis and there is no field variation in this direction. The structure is excite with TM field which means that:

$$H_z = 0 \tag{1}$$

$$H_x = 0 \tag{2}$$

$$E_y = 0 \tag{3}$$

2 Scattering of electromagnetic field on a dielectric rod of cylindrical cross-section

The dielectric rod is placed coaxially to the z axis. In this case the computational domain is divided into two regions (Fig. 1). The first region is a vacuum and the second is a dielectric. The E_z field gives precise information about the fields in these two regions and fulfills the Helmholtz equation. The excitation can be defined as:

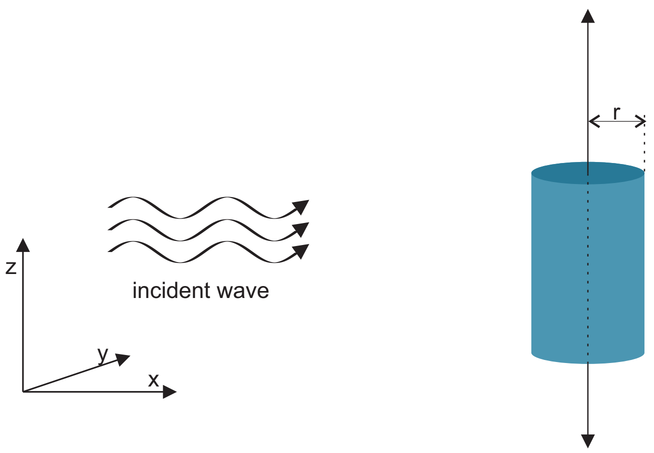

Figure 1: The considered structure (dielectric rod of circular cross section)

$$E_z^i = \sum_{m=-M}^{M} c_m J_m(\kappa_1 \rho) e^{jm\phi} \tag{4}$$

where $\kappa_i = \omega\sqrt{\mu_i \varepsilon_i}$. The scattered field can be expressed:

$$E_z^s = \sum_{m=-M}^{M} a_m H_m^{(2)}(\kappa_1 \rho) e^{jm\phi} \tag{5}$$

The electric fields inside and outside the rod can be then written as

$$E_z^{out} = \sum_{m=-M}^{M} (c_m J_m(\kappa_1 \rho) + a_m H_m^{(2)}(\kappa_1 \rho)) e^{jm\phi} \tag{6}$$

and

$$E_z^{in} = \sum_{m=-M}^{M} b_m J_m(\kappa_2 \rho) e^{jm\phi} \tag{7}$$

The corresponding magnetic fields are

$$H_\phi^{out} = \frac{\kappa_1}{j\omega\mu_1} \sum_{m=-M}^{M} (c_m J_m'(\kappa_1 \rho) + a_m H_m'^{(2)}(\kappa_1 \rho)) e^{jm\phi} \tag{8}$$

and

$$H_\phi^{in} = \frac{\kappa_2}{j\omega\mu_2} \sum_{m=-M}^{M} b_m J_m'(\kappa_2 \rho) e^{jm\phi} \tag{9}$$

where c_m is an excitation coefficient and a_m and b_m are unknown coefficients. By using boundary conditions at $\rho = r$ and $\phi \in [0, 2\pi]$ the unknown coefficients can be found (tangential components of the fields outside and inside the rod must be equal):

$$a_m = \frac{\kappa_1 J_m(\kappa_2 r) J_m'(\kappa_1 r) - \kappa_2 J_m(\kappa_1 r) J_m'(\kappa_2 r)}{\kappa_2 H_m^{(2)}(\kappa_1 r) J_m'(\kappa_2 r) - \kappa_1 J_m(\kappa_2 r) H_m'^{(2)}(\kappa_1 r)} c_m \tag{10}$$

$$b_m = \frac{\kappa_1 H_m'^{(2)}(\kappa_1 r) a_m + \kappa_1 J_m'(\kappa_1 r) c_m}{\kappa_2 J_m'(\kappa_2 r)} \tag{11}$$

In this section we consider only the plane wave as an excitation but the equations are general. For the plane wave $c_m = j^{-m}$.

In Fig. 2 the scattering characteristics for three different radiuses of the rod are presented ($r = 38,2mm$, $r = 9,5mm$ and $r = 2,4mm$ respecively; where frequency $f = 5GHz$ and $\varepsilon_{r2} = 5$). In order to determine the optimal value of M (to obtain the complete set of the function $M \to \infty$) a brief analysis of convergence is presented. In Fig. 3 the scattering characteristics for different numbers of basis functions is shown. Moreover, the scattered field coefficients a_m are presented in Fig. 4. According to the results $M = 15$ is assumed in the subsequent part of the report.

3 Z matrix

To calculate **Z** matrix, the domain need to be divide into two regions (I and II) as in the Fig. 5. There is an artificial layer at the boundary of the regions. On this layer the **Z** matrix can be defined as a relation between the total electric field and the total magnetic field. The **Z** matrix unambiguously defines the surrounded structure. After calculation its inside can be treated as a black box. In this approach the fields on the layer can be written as:

$$E_z^{tot}(\rho = R, \phi) = \sum_{m=-M}^{M} (c_m J_m(\kappa_1 R) + a_m H_m^{(2)}(\kappa_1 R)) e^{jm\phi} \tag{12}$$

and

$$H_\phi^{tot}(\rho = R, \phi) = \frac{\kappa_1}{j\omega\mu_1} \sum_{m=-M}^{M} (c_m J_m'(\kappa_1 R) + a_m H_m'^{(2)}(\kappa_1 R)) e^{jm\phi} \tag{13}$$

The **Z** matrix can be written as a relation between the coefficients of the electric and magnetic fields expansion in terms of $e^{jm\phi}$:

$$\mathbf{d} = \mathbf{Z}\mathbf{e} \tag{14}$$

where **d** nad **e** are vertical vectors composed of the elements (m=-M,..,M):

$$d_m = c_m J_m(\kappa_1 R) + a_m H_m^{(2)}(\kappa_1 R) \tag{15}$$

$$e_m = \frac{\kappa_1}{j\omega\mu_1} (c_m J_m'(\kappa_1 R) + a_m H_m'^{(2)}(\kappa_1 R)) \tag{16}$$

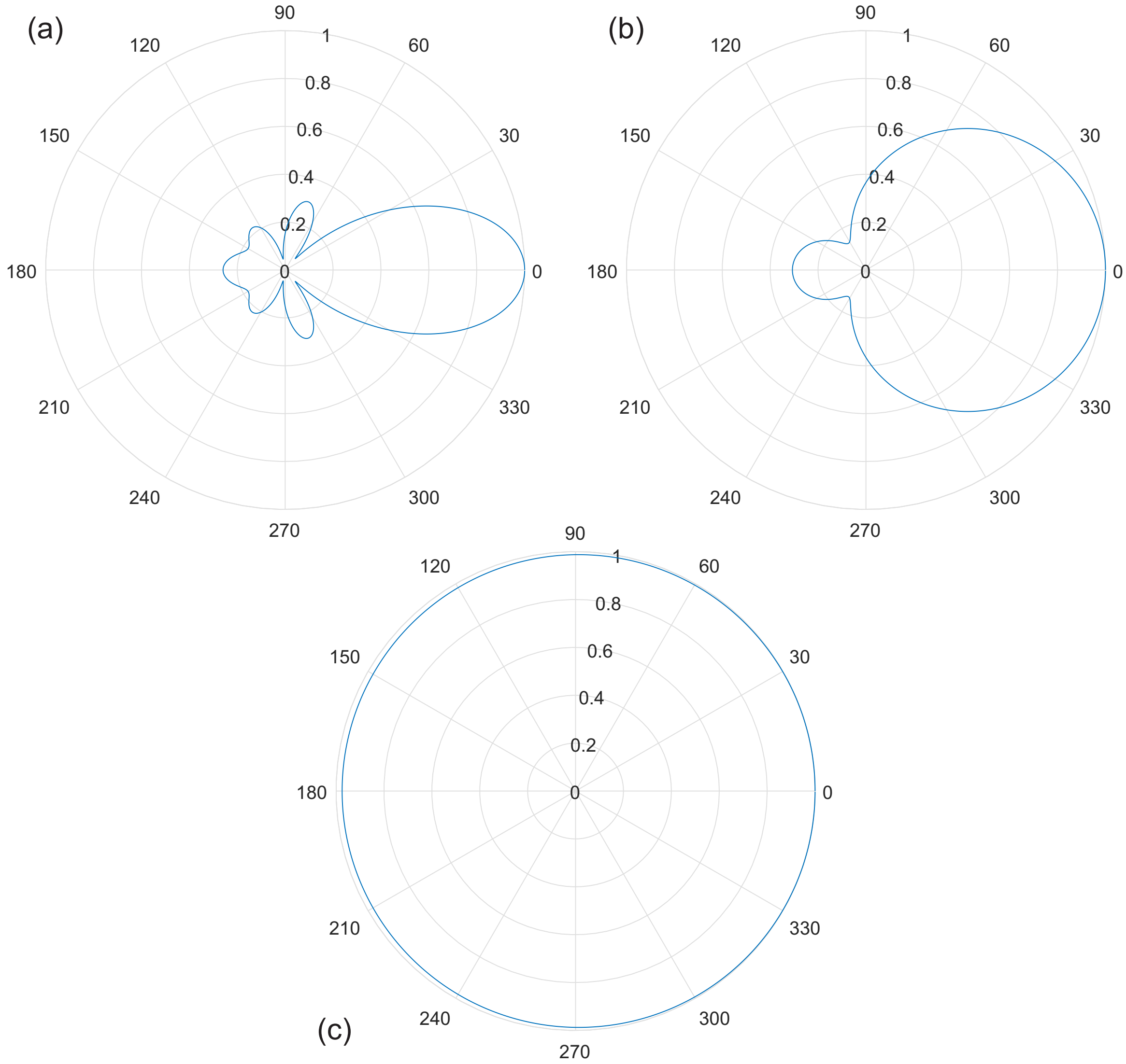

Figure 2: The scattering characteristic a)$r = 38,2mm$ b)$r = 9,5mm$ c) $r = 2,4mm$

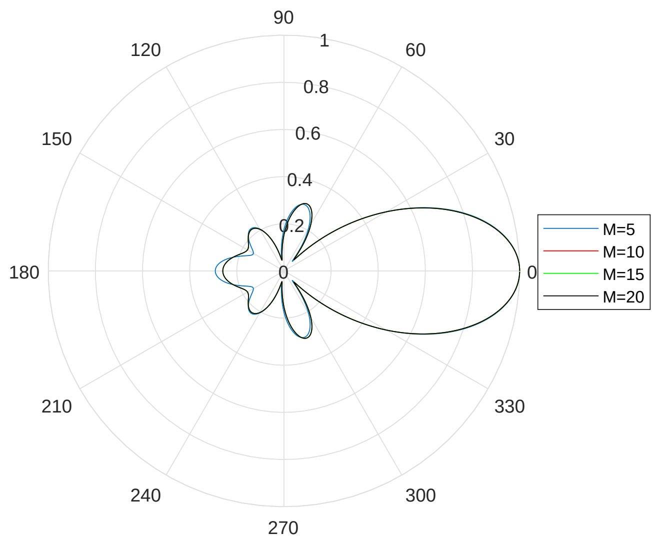

Figure 3: The scattering characteristics for different values of M

The exact form of the impedance matrix can be found by combination of (12), (13) and (10):

$$\mathbf{Z} = (\mathbf{HT} + \mathbf{J})(\widehat{\mathbf{H}}\mathbf{T} + \widehat{\mathbf{J}})^{-1} \tag{17}$$

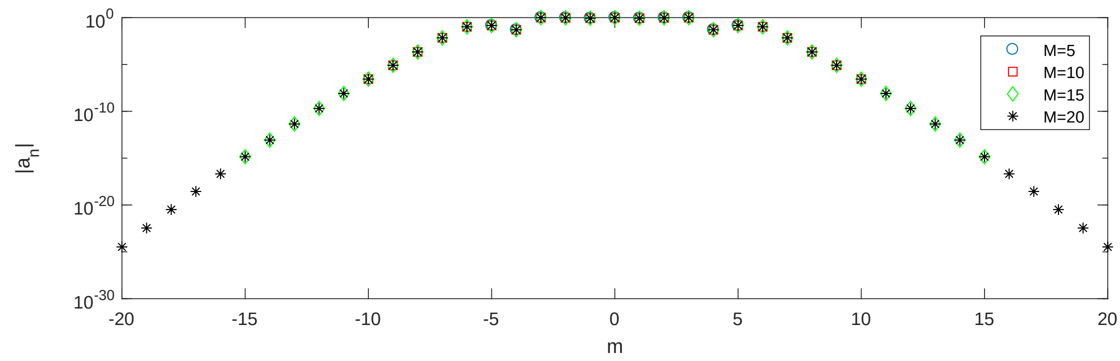

Figure 4: Coefficients of the scattered field

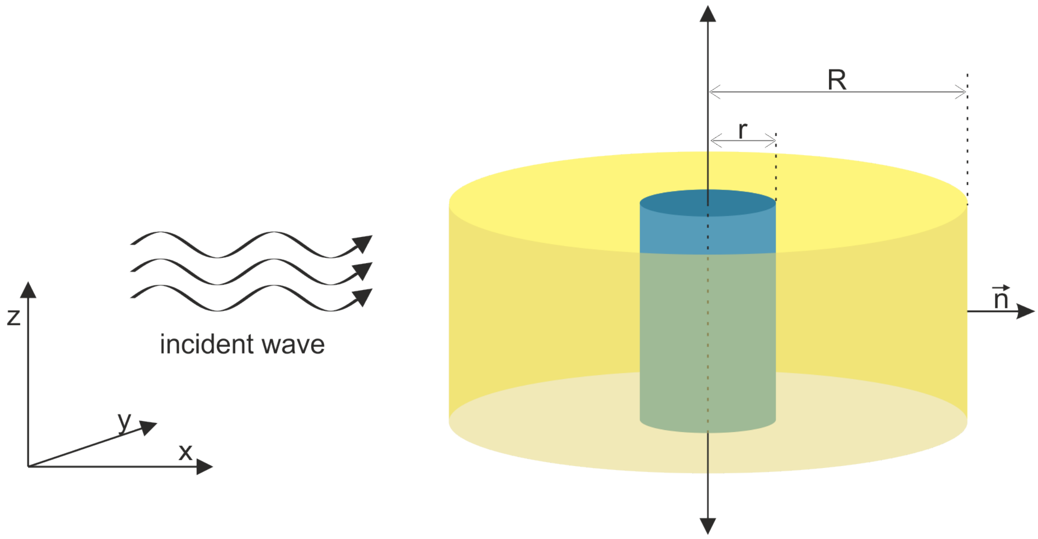

Figure 5: Domain decomposition used in Z matrix definition

where $\mathbf{T}$ matrix is a matrix of transformation between coefficients a_m and c_m (diagonal matrix where coefficients on the diagonal can be found using the equation (10)), $\mathbf{H}= diag(H_{-M}^{(2)}(\kappa_1 R), ..., H_M^{(2)}(\kappa_1 R))$ is a diagonal matrix with Hankel function, $\mathbf{J} = diag(J_{-M}(\kappa_1 R), ..., J_M(\kappa_1 R))$, $\widehat{\mathbf{J}} = diag(J'_{-M}(\kappa_1 R), ..., J'_M(\kappa_1 R))$ and $\widehat{\mathbf{H}} = diag(H'^{(2)}_{-M}(\kappa_1 R), ..., H'^{(2)}_M(\kappa_1 R))$.

For the same cases as in the previous section the scattering characteristics are shown in Fig. 6. The results are exactly the same as in the direct approach (see Fig 6).

4 Z matrix estimation using FEM

The problem of scattering on any cylindrical object homogeneous in the direction of the z axis for TM modes is described by Helmholtz equation (we also assume that there is no field variation in the z direction):

$$\vec{\nabla}_t^2 E_z(x, y) + k^2 E_z(x, y) = 0 \tag{18}$$

To solve this equation, the formula have to be multiplied by testing function $F(x, y)$ and integrated over the computational domain S (region I in Fig. 5).

$$\iint_S F(x, y)\vec{\nabla}_t^2 E_z(x, y)\ ds + \iint_S F(x, y) k^2 E_z(x, y)\ ds = 0 \tag{19}$$

By using Green's theorem the equation (19) can be expressed as:

$$\oint_L F(x, y)\vec{\nabla}_t E_z \circ \vec{n}\ dl - \iint_S \vec{\nabla}_t F(x, y) \circ \vec{\nabla}_t E_z(x, y)\ ds + \iint_S F(x, y) k^2 E_z(x, y)\ ds = 0 \tag{20}$$

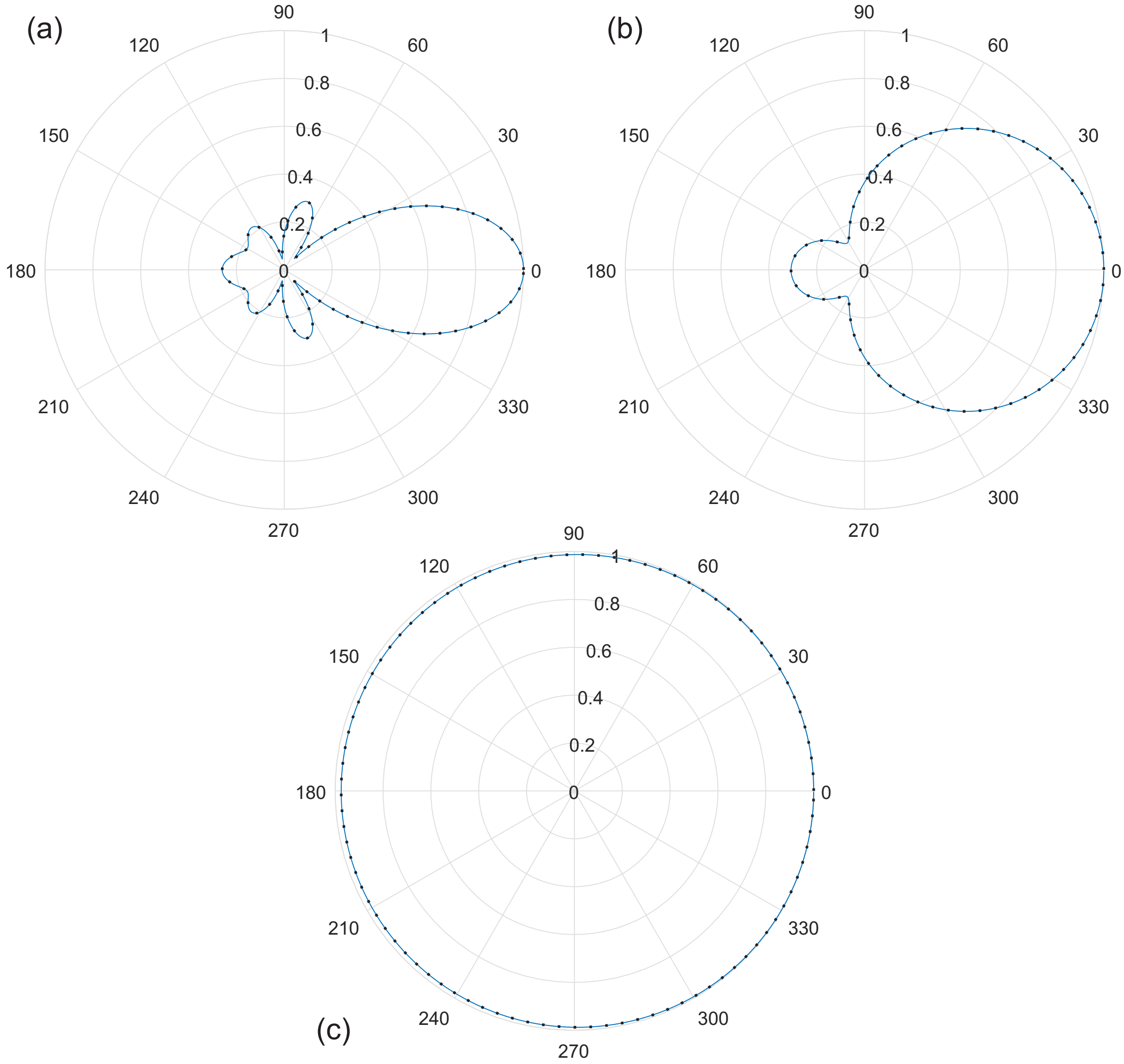

Figure 6: The scattering characteristic a) $r = 38,2mm$ b) $r = 9,5mm$ c) $r = 2,4mm$ (solid line - analytical results; dotted line - $\mathbf{Z}$ matrix is used)

where L is the boundary of the computational domain - circle of radius R (see Fig. 5) and the unit vector $\vec{n} = \vec{i}_\rho$. The dependence between E_z and H_t from Maxwell equation can be written as:

$$\vec{\nabla}_t E_z = -j\omega\mu \vec{i}_z \times \vec{H}_t \tag{21}$$

If the equation (21) and equation (20) are combined then the problem can be rewritten:

$$\iint\limits_S \vec{\nabla}_t F(x,y) \circ \vec{\nabla}_t E_z(x,y)\, ds - \iint\limits_S F(x,y) k^2 E_z(x,y)\, ds + j\omega\mu \oint\limits_L F(x,y)(\vec{i}_z \times \vec{H}_t) \circ \vec{n}\, dl = 0 \tag{22}$$

It can be also written:

$$\iint\limits_S \vec{\nabla}_t F(x,y) \circ \vec{\nabla}_t E_z(x,y)\, ds - \iint\limits_S F(x,y) k^2 E_z(x,y)\, ds - j\omega\mu \oint\limits_L F(x,y)\vec{i}_z \circ (\vec{n} \times \vec{H}_t)\, dl = 0 \tag{23}$$

The magnetic field can be treated as an excitation on the boundary L.

$$\vec{H}_t = \sum_{m=-M}^{M} I_m \vec{h}_{tm} \tag{24}$$

where (in the considered scattering problem) we assume $\vec{h_{tm}} = e^{jm\phi}\vec{i}_\phi$. Using equation (24) gives us formula:

$$\iint\limits_S \vec{\nabla}_t F(x,y) \circ \vec{\nabla}_t E_z(x,y)\ ds - \iint\limits_S F(x,y) k^2 E_z(x,y)\ ds - j\omega\mu \sum_{m=-M}^{M} I_m \oint\limits_L F(x,y)\vec{i}_z \circ (\vec{n} \times \vec{h}_{tm})\ dl = 0 \tag{25}$$

The electric field is expressed by the basis functions

$$E_z = \sum_{n=1}^{N}\sum_{i=3}^{3} \Psi_{(i)}^{[n]} \alpha_{(i)}^{[n]} \tag{26}$$

and then Galerkin method can be applied

$$F(x,y) = \alpha_{(j)}^{[q]} \tag{27}$$

Hence

$$\sum_{i=1}^{3} \iint\limits_{S^{[q]}} \vec{\nabla}_t \alpha_{(j)}^{[q]} \circ \vec{\nabla}_t \alpha_{(i)}^{[q]}\ ds - \sum_{i=1}^{3} \Psi_{(i)}^{[q]} \iint\limits_{S^{[q]}} \alpha_{(j)}^{[q]} k^2 \alpha_{(i)}^{[q]}\ ds - j\omega\mu \sum_{m=-M}^{M} I_m \oint\limits_{L\cap L^{[q]}} \alpha_{(j)}^{[q]} \vec{i}_z \circ (\vec{n} \times \vec{h}_{tm})\ dl = 0 \tag{28}$$

The equation can be written in matrix notation:

$$C^{[q]}\Psi^{[q]} - k^2 T^{[q]}\Psi^{[q]} = -j\omega\mu B^{[q]} I \tag{29}$$

where

$$B^{[q]} = \left[B_{-M}^{[q]}, \cdots, B_M^{[q]}\right] \tag{30}$$

$$B^{[q]} = \begin{bmatrix} \int\limits_{L\cap L^{[q]}} \alpha_1^{[q]}\vec{i}_z \circ (\vec{n} \times \vec{h}_{t(-M)})\ dl & \int\limits_{L\cap L^{[q]}} \alpha_1^{[q]}\vec{i}_z \circ (\vec{n} \times \vec{h}_{t(-M+1)})\ dl & \cdots & \int\limits_{L\cap L^{[q]}} \alpha_1^{[q]}\vec{i}_z \circ (\vec{n} \times \vec{h}_{tM})\ dl \\ \int\limits_{L\cap L^{[q]}} \alpha_2^{[q]}\vec{i}_z \circ (\vec{n} \times \vec{h}_{t(-M)})\ dl & \int\limits_{L\cap L^{[q]}} \alpha_2^{[q]}\vec{i}_z \circ (\vec{n} \times \vec{h}_{t(-M+1)})\ dl & \cdots & \int\limits_{L\cap L^{[q]}} \alpha_2^{[q]}\vec{i}_z \circ (\vec{n} \times \vec{h}_{tM})\ dl \\ \int\limits_{L\cap L^{[q]}} \alpha_3^{[q]}\vec{i}_z \circ (\vec{n} \times \vec{h}_{t(-M)})\ dl & \int\limits_{L\cap L^{[q]}} \alpha_3^{[q]}\vec{i}_z \circ (\vec{n} \times \vec{h}_{t(-M+1)})\ dl & \cdots & \int\limits_{L\cap L^{[q]}} \alpha_3^{[q]}\vec{i}_z \circ (\vec{n} \times \vec{h}_{tM})\ dl \end{bmatrix} \tag{31}$$

and

$$I = \begin{bmatrix} I_{-M} \\ I_{-M+1} \\ \vdots \\ I_M \end{bmatrix} \tag{32}$$

The equation (29) can be rewrite in global form.

$$C\Psi - k^2 T\Psi = -j\omega\mu B I \tag{33}$$

or simply by

$$G\Psi = -j\omega\mu B I \tag{34}$$

The electric field can be expressed by:

$$E_z = \sum_{s=-M}^{M} V_s e_{zs} \tag{35}$$

where we assume the basis functions as $e_{zs} = e^{jm\phi}$. By simple projection with the use of functions $\vec{n} \times h_{tm}$ we get:

$$\int\limits_L E_z \vec{i}_z \circ (\vec{n} \times \vec{h}_{tm})\ dl = \sum_{s=-M}^{M} V_s \int\limits_L e_{zs}\vec{i}_z \circ (\vec{n} \times \vec{h}_{tm})\ dl \tag{36}$$

Table 2: Error of $\mathbf{Z^{FEM}}$ matrix defined by $\mathbf{Err} = \mathbf{Z} - \mathbf{Z^{FEM}}$ in function of maximal mesh edge length

h_{max}	$\max\limits_{i,j}[\lvert Err \rvert]_{i,j} / \max\limits_{i,j}[\lvert Z \rvert]_{i,j} 100\%$	$\frac{\lambda}{h_{max}}$
0.003	43.5	8.9381
0.002	41.6	13.4071
0.001	11.4	26.8143

Then using (26)

$$\sum_{q=1}^{N} \sum_{i=1}^{3} \Psi_{(i)}^{[q]} \int\limits_{L \cap L^{[q]}} \alpha_{(i)}^{[q]} \vec{i}_z \circ (\vec{n} \times \vec{h}_{tm}) \, dl = \sum_{s=-M}^{M} V_s \int\limits_{L} e_{zs}^{p} \vec{i}_z \circ (\vec{n} \times \vec{h}_{tm}) \, dl \tag{37}$$

Finally

$$\sum_{q=1}^{N} (B_m^{[q]})^T \Psi^{[q]} = \Delta_m V \tag{38}$$

or simply

$$B^T \Psi = \Delta V \tag{39}$$

From (34) we get

$$\Psi = -j\omega\mu G^{-1} B I \tag{40}$$

Then combining both above equations we get:

$$V = j\omega\mu \Delta^{-1} B^T G^{-1} B I \tag{41}$$

Finally the $\mathbf{Z^{FEM}}$ matrix can be expressed as:

$$Z^{FEM} = j\omega\mu \Delta^{-1} B^T G^{-1} B \tag{42}$$

where Δ is identity matrix multiplied by $2\pi R$.

5 Hybrid method

In the presented approach finite element method and analitycal approach are combined. Using the $\mathbf{Z}$ matrix, the far field can be obtained for any incident field by simple reformulation of (17).

$$\begin{bmatrix} a_{-M} \\ a_{-M+1} \\ \vdots \\ a_m \end{bmatrix} = (\mathbf{H} - \mathbf{Z^{FEM}} \widehat{\mathbf{H}})^{\mathbf{-1}} (\mathbf{Z^{FEM}} \widehat{\mathbf{J}} - \mathbf{J}) \begin{bmatrix} c_{-M} \\ c_{-M+1} \\ \vdots \\ c_m \end{bmatrix} \tag{43}$$

The result of combining analytical and finite element method (FEM) approaches are shown in Fig 7. The characteristics well agree with the analytical result for finer meshes. In table 2 error between analytical $\mathbf{Z}$ matrix and $\mathbf{Z^{FEM}}$ for different meshes are collected.

6 Conclusions and further development

The proposed approach has been verified for a simple structure of circular cross section (due to exact analytical results). However it can by applied for obstacles of any cross section shape. The numerical tests for different structures will be performed in the near future.

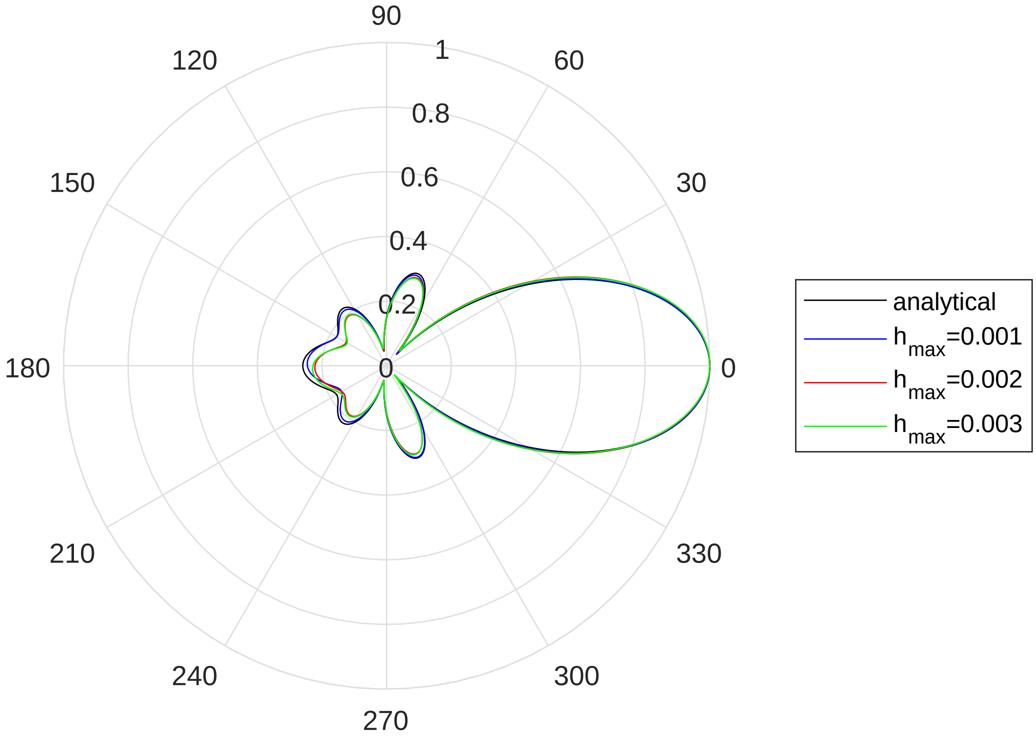

Figure 7: Characteristics for different values of maximal triangle edge length h_{max} (where λ is the wavelength inside the dielectric)