Electromagnetic Design of
flexIble SensOrs

Report 3 - FEM-MM Formulation for Scattering Electromagnetic Field Computation Enhanced by MOR

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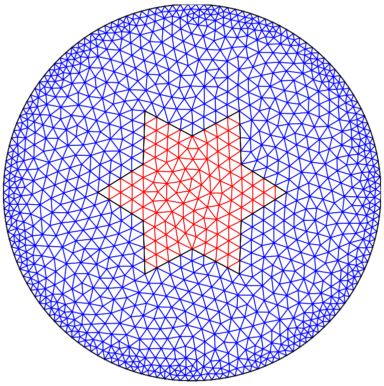
1. Formulation of the problem



Analyzed structure is star. Number of modes at the boundary is defined with the number M, but the final size is $m = -M : M \times 2$. For example if $M = 3$ then $m = 14$. Frequency band is defined with number of frequency points n_f, minimum frequency f_{min} and maximum frequency f_{max}. The original model system of equation size is N and the reduced model size is $n = n_Q \cdot m$ where n_Q is number of subsequent block moments at subsequent expansion points.

The goal of this rep is to compare RBM, SAPOR and GM-MOR algorithms application to scattering problem formulation based on the hybrid FEM and Mode matching method.

1.1. Problem construction

System of equations:	$\mathbf{G}\Psi = -j\omega\mu\mathbf{B}$
System matrix:	$\mathbf{G} \in \mathbb{C}^{N\times N}$
Right side vector:	$\mathbf{B} \in \mathbb{C}^{N\times m}$
Solution vector:	$\mathbf{\Psi} \in \mathbb{C}^{N\times m}$
Reduced solution vector:	$\mathbf{\Psi}_r \in \mathbb{C}^{n\times m}$
Approximated solution vector:	$\mathbf{\Psi} \approx \mathbf{Q}\mathbf{\Psi}_r$
Local real error:	$\mathrm{norm}(\mathbf{Q}\mathbf{\Psi}_r - \mathbf{\Psi})/\mathrm{norm}(\mathbf{\Psi})$

Subsequent projection basis columns are constructed using block moments of Ψ expansion point at specific frequencies to represent the evolution of electromagnetic field as a function of frequency. Next expansion points are chosen using error estimator.

1.2. Scattered field

Original field:	$outE$
MOR field:	$outE_r$
Far field error:	$\max(outE - outE_r)$

2. Computational optimization

The block matrix $\mathbf{G}$ is defined as:

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_{zz} & \mathbf{G}_{zt} \\ \mathbf{G}_{tz} & \mathbf{G}_{tt} \end{bmatrix}, \tag{1}$$

where inner components depend on frequency variables $k_0 = 2\pi f$ and $\beta = jk_0 cos(\theta)$, where θ is a angle of falling wave. Taking into account three frequency dependent cases and constructing of block matrices results in:

$$\begin{aligned} \mathbf{G}_{zz}(k_0^2) &= \mathbf{C}_s + k_0^2 \mathbf{T}_s, \\ \mathbf{G}_{zt}(\beta) &= \beta \mathbf{G}_{zt}, \\ \mathbf{G}_{tz}(\beta) &= \beta \mathbf{G}_{tz}, \\ \mathbf{G}_{tt}(k_0^2, \beta^2) &= \mathbf{C}_v + k_0^2 \mathbf{T}_v + \beta^2 \mathbf{G}_v. \end{aligned} \tag{2}$$

Consider $cos(\theta)$ as constant and introduce $s = j\omega/c$. After that we can introduce $cos\theta$ as a component of system matrices to obtain following formulation of problem:

$$\begin{aligned} \mathbf{G}\mathbf{\Psi} &= -j\omega\mu \mathbf{B}, \\ \mathbf{G} &= s^2 \mathbf{T} + s\mathbf{G}_{temp} + \mathbf{C}. \end{aligned} \tag{3}$$

The construction of matrices $\mathbf{C}$, $\mathbf{T}$ and $\mathbf{G}_{temp}$ is organized as follows:

$$\begin{aligned} \mathbf{C} &= \begin{bmatrix} \mathbf{C}_s & 0 \\ 0 & \mathbf{C}_v \end{bmatrix}, \\ \mathbf{T} &= \begin{bmatrix} \mathbf{T}_s & 0 \\ 0 & \mathbf{T}_v + \mathbf{G}_v \cdot cos^2(\theta) \end{bmatrix}, \\ \mathbf{G}_{temp} &= \begin{bmatrix} 0 & \mathbf{G}_{zt} \cdot cos(\theta) \\ \mathbf{G}_{tz} \cdot cos(\theta) & 0 \end{bmatrix}. \end{aligned} \tag{4}$$

At this moment we obtained frequency independent matrices with the same order $N \times N$. We can now introduce error estimator given by:

$$\begin{aligned} \mathbf{e}(f) &= \text{norm}(\mathbf{B}^H/\text{norm}(\mathbf{B}) \cdot (\mathbf{G}\mathbf{Q}\Psi + j\omega\mu\mathbf{B})/\text{norm}(j\omega\mu\mathbf{B})) \\ \mathbf{e}(f) &= \text{norm}(\mathbf{B}^H\mathbf{G}\mathbf{Q}\Psi + j\omega\mu\mathbf{B}^H\mathbf{B})/\text{norm}(j\omega\mu)/\text{norm}(\mathbf{B})^2) \end{aligned} \tag{5}$$

Note that $\mathbf{B}$ is frequency independent, so $\mathbf{B}^H\mathbf{C}\mathbf{Q}$, $\mathbf{B}^H\mathbf{T}\mathbf{Q}$, $\mathbf{B}^H\mathbf{G}_{temp}\mathbf{Q}$, $\mathbf{B}^H\mathbf{B}$ and norm($\mathbf{B}$) can be computed before frequency sweep, which makes this estimator very fast operation on low order $n \times n$ matrices. To evaluate the action of proposed error estimator, we have introduced the real local error given by:

$$\text{norm}(|\mathbf{Q}\Psi_r - \Psi_{ref}|)/\text{norm}(\Psi_{ref}).$$

We have obtained good correlation between estimator and local real error. The results will be presented in next sections. Additionally we have introduced impedance matrix error estimator:

$$\text{norm}(|\mathbf{Z}_r - \mathbf{Z}_{ref}|)/\text{norm}(\mathbf{Z}_{ref}),$$

but similarly as in local real error, the reference is not available at the moment of random structure computation. Both definitions are used to judge validity of error estimator.

3. Impedance matrix and solution vector Ψ observation

Impedance matrix and solution vector Ψ have the same core elements for M=3 at M=10. It means that removing outer and inner columns of matrix for M=10, results in obtaining matrix for M=3. We can observe this occurence at fig. 1. Additionally the element error relies on tolerance of simulation. The Z matrix error is lower in contrary to Ψ error, which can be seen at report attachment.

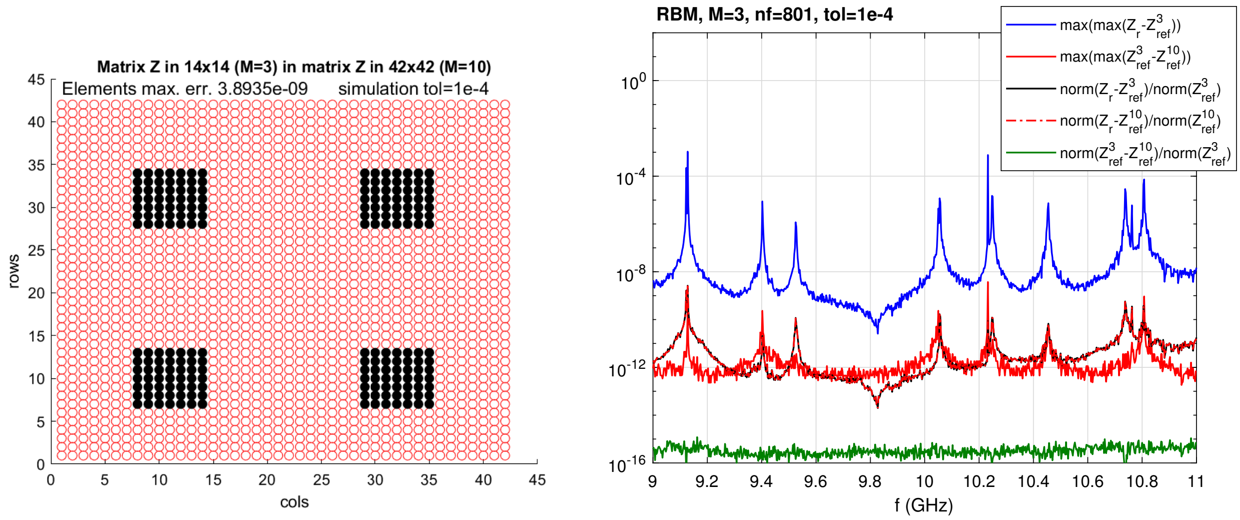


Figure 1: On the left: Black elements are elements obtained via RBM case and red circles are obtained via reference. Comparison made at frequency vector present in projection basis.
On the right: Comparison of Z matrix elements. Reference matrices for M=3 and M=10 are compared with obtained via RBM for M=3.

Blue line on the right plot corresponds to error between maximum absolute values of difference between reference and RBM matrices for M=3. Red line is difference between reference matrices for M=3 and M=10 reduced to M=3. Black, green and dashed red lines are error functions introduced in section 1.1.

Following feature can be used to estimate real error based on far field. The estimation cost will be slightly less than final sweep time.

4. PARDISO Solver optimization

This operation consists of five steps:

1. Analysis
2. Numerical factorization
3. Solve
4. Release internal memory for L and U
5. Release all internal memory for all matrices

For frequency sweep operation, step 1. should be made once, because position of nonzero elements is not changing. Steps 2-4 are performed for every frequency point, but iparm and pt parameters obtained at step 1 must be refreshed every time we start step 2. At the end of program, user should release all internal memory using step 5. Solution times obtained for UMFPack and PARDISO procedures are presented at table below.

Comparison have been made between UMF and PARDISO and is presented at 2. The difference is at machine precision level until tolerance at 1e-10. Then PARDISO has an advantage over UMF what can be seen at Ψ error and estimate lines.

Table 1: Real symmetric N=200 232 unknowns, No. of iterations=11, nnz=0.18%

	Full time	Step 1	Step 2	Steps 3-5	Max. Err.	Norm(x)
UMFPack (Ref)	16.3 s	-	-	-	-	-
PARDISO	8.4 s	2.15 s	5.3 s	0.9 s	5.1e-9	284

Table 2: Complex symmetric N=45 000 unknowns, No. of iterations=11, nnz=0.048%

	Full time	Step 1	Step 2	Steps 3-5	Max. Err.	Norm(x)
UMFPack (Ref)	161 s	-	-	-	-	-
PARDISO	3.3 s	0.4 s	2.5 s	0.4 s	7.1e-5	122.7

Table 3: Hermitian (analyzed type) N=177981 unknowns, No. of iterations=11, nnz=0.013%

	Full time	Step 1	Step 2	Steps 3-5	Max. Err.	Norm(x)
UMFPack (Ref)	55.2 s	-	-	-	-	-
PARDISO	5.8 s	1.1 s	3.8 s	0.9 s	4.6e-14	1.91

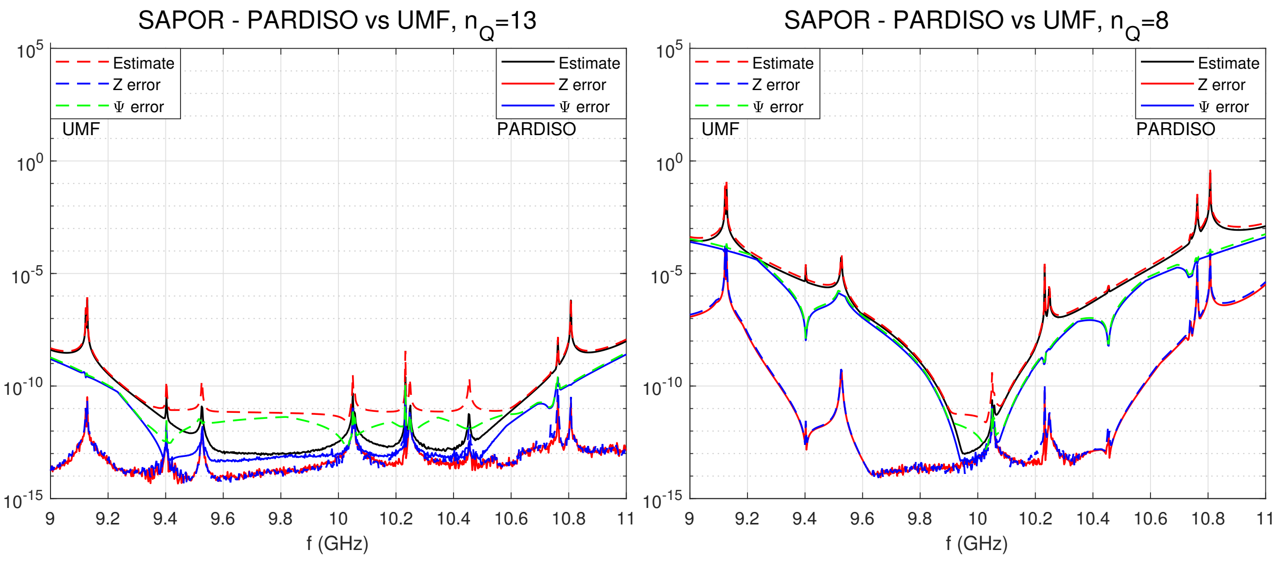


Figure 2: Comparison of PARDISO and UMFPack solvers at steps: 13 and 8.

5. Numerical Experiments

Experiments were performed with three input variables number of frequency subranges NF, number of Hankel functions M and maximum number of block moments Q_{max}. Tests does not include mesh and matrix generation. Two cases were taken into account, one with ca. **16000** and second with ca. **177000** unknowns. Blue row is slowest case, red is fastest with Gram-Schmidt algorithm, and light red is fastest overall (RBM with SVD algorithm).

5.1. Band 9-11 GHz, 801 points, tolerance 1e-4, N=15729

Table 4: Analysis result for M =**3**

	$n(n_Q)$	Full time	MOR time	Sweep Time	Speedup	Max. Est.
Original formulation	15729	49.2 s	0 s	49.2 s	1	-
SAPOR	182 (13)	16.5 s	8.6 s	7.9 s	2.98	6.9e-7
GM-MOR						
$NF=10$, $Q_{max}=10$	140 (10)	**15 s**	**7.3 s**	**7.7 s**	3.28	7.9e-5
$NF=20$, $Q_{max}=5$	151 (11)	17.2 s	9.4 s	**7.7 s**	2.86	**5.9e-7**
$NF=40$, $Q_{max}=2$	154 (11)	17.6 s	9.8 s	7.8 s	2.79	2.6e-6
RBM	**140** (10)	15.1 s	7.4 s	**7.7 s**	3.26	8.1e-6
RBM (SVD)	140 (10)	13.3 s	5.6 s	7.6 s	3.7	9.9e-6

Table 5: Analysis result for M =**5**

	$n(n_Q)$	Full time	MOR time	Sweep Time	Speedup	Max. Est.
Original formulation	15729	61.6 s	0 s	61.6 s	1	-
SAPOR	242 (11)	**22.8 s**	**11.2 s**	11.5 s	2.7	2.2e-5
GM-MOR						
$NF=10$, $Q_{max}=10$	**197** (9)	24.2 s	13 s	**11.2 s**	2.55	6.4e-6
$NF=20$, $Q_{max}=5$	216 (10)	27.8 s	16.4 s	11.4 s	2.21	**3.2e-6**
$NF=40$, $Q_{max}=2$	198 (9)	25.9 s	14.7 s	**11.2 s**	2.38	6.1e-5
RBM	198 (9)	23.4 s	12 s	11.4 s	2.63	9.9e-5
RBM (SVD)	220 (10)	21.6 s	9.9 s	11.7 s	2.85	5.1e-7

Table 6: Analysis result for M =**10**

	$n(n_Q)$	Full time	MOR time	Sweep Time	Speedup	Max. Est.
Original formulation	15729	95.9 s	0 s	95.9 s	1	-
SAPOR	378 (9)	52.9 s	23.3 s	29.6 s	1.81	4.2e-5
GM-MOR						
$NF=10$, $Q_{max}=10$	378 (9)	70.7 s	41 s	29.6 s	1.36	4.9e-5
$NF=20$, $Q_{max}=5$	373 (9)	78.1 s	48.4 s	29.7 s	1.23	6.9e-5
$NF=40$, $Q_{max}=2$	373 (9)	79.9 s	50.3 s	29.6 s	1.2	**1.8e-5**
RBM	**294** (7)	**50.6 s**	**22.3 s**	**28.2 s**	1.89	3.2e-5
RBM (SVD)	294 (7)	42.3 s	13.2 s	29.1 s	2.27	3.2e-5

5.2. Band 7-13 GHz, 1201 points, tolerance 1e-4, N=15729

Table 7: Analysis result for M =**3**

	$n(n_Q)$	Full time	MOR time	Sweep Time	Speedup	Max. Est.
Original formulation	15729	73.8 s	0 s	73.8 s	1	-
SAPOR	280 (20)	44.7 s	30.3 s	14.4 s	1.65	3.1e-5
GM-MOR						
$NF = 10$, $Q_{max} = 10$	250 (18)	39.1 s	26.3 s	12.8 s	1.88	2.9e-5
$NF = 20$, $Q_{max} = 5$	235 (17)	37.2 s	24.5 s	12.7 s	1.98	**3.8e-6**
$NF = 40$, $Q_{max} = 2$	**224** (16)	**35.9 s**	23.3 s	**12.6 s**	2.06	9.4e-5
RBM	**224** (16)	**35.8 s**	**23.1 s**	**12.6 s**	2.06	1.2e-5
RBM (SVD)	224 (16)	32.9 s	19.7 s	13.1 s	2.24	1.2e-5

Table 8: Analysis result for M =**5**

	$n(n_Q)$	Full time	MOR time	Sweep Time	Speedup	Max. Est.
Original formulation	15729	92.4 s	0 s	92.4 s	1	-
SAPOR	374 (17)	71 s	49.1 s	21.9 s	1.3	3.1e-5
GM-MOR						
$NF = 10$, $Q_{max} = 10$	330 (15)	59 s	38.9 s	20.1 s	1.57	5.6e-5
$NF = 20$, $Q_{max} = 5$	329 (15)	69.1 s	48.9 s	20.2 s	1.34	**4.8e-6**
$NF = 40$, $Q_{max} = 2$	308 (14)	63.8 s	44 s	19.7 s	1.45	1.9e-5
RBM	**286** (13)	**51.8 s**	**32.6 s**	**19.2 s**	1.78	3.9e-5
RBM (SVD)	286 (13)	46.1 s	26.5 s	19.6 s	2	1.2e-5

Table 9: Analysis result for M =**10**

	$n(n_Q)$	Full time	MOR time	Sweep Time	Speedup	Max. Est.
Original formulation	15729	143.8 s	0 s	143.8 s	1	-
SAPOR	588 (14)	**155 s**	**100 s**	55.1 s	0.93	4.9e-5
GM-MOR						
$NF = 10$, $Q_{max} = 10$	561 (14)	178 s	125 s	53.4 s	0.81	3.4e-5
$NF = 20$, $Q_{max} = 5$	**539 (13)**	167 s	115 s	**51.9 s**	0.86	5.7e-6
$NF = 40$, $Q_{max} = 2$	578 (14)	219 s	165 s	54.5 s	0.66	**4.1e-6**
RBM	546 (13)	170 s	117 s	53.2 s	0.85	5.7e-5
RBM (SVD)	504 (12)	122 s	70.7 s	51.5 s	1.18	6.5e-5

5.3. Band 9-11 GHz, 801 points, tolerance 1e-4, N=177981

Table 10: Analysis result for M =**3**

	$n(n_Q)$	Full time	MOR time	Sweep Time	Speedup	Max. Est.
Original formulation	177981	562 s	—	562 s	—	—
SAPOR	154 (11)	47.1 s	39.4 s	7.7 s	11.9	6e-5
GM-MOR						
$NF = 10$, $Q_{max} = 10$	151 (11)	95 s	87.3 s	7.7 s	5.92	2.1e-7
$NF = 20$, $Q_{max} = 5$	140 (10)	80.2 s	72.6 s	7.6 s	7	1.1e-5
$NF = 40$, $Q_{max} = 2$	—	—	—	—	—	—
RBM	140 (10)	65.3 s	57.6 s	7.6 s	8.61	4.6e-5
RBM (SVD)	140 (10)	44.4 s	36.6 s	7.7 s	12.66	4.6e-5

Table 11: Analysis result for M =**5**

	$n(n_Q)$	Full time	MOR time	Sweep Time	Speedup	Max. Est.
Original formulation	177981	650 s	—	650 s	—	—
SAPOR	220 (10)	70.5 s	59.2 s	11.3 s	9.22	4.3e-5
GM-MOR						
$NF = 10$, $Q_{max} = 10$	176 (8)	105.6 s	94.5 s	11 s	6.15	3.8e-5
$NF = 20$, $Q_{max} = 5$	198 (9)	147.2 s	135.9 s	11.3 s	4.41	6.9e-6
$NF = 40$, $Q_{max} = 2$	198 (9)	158.5 s	147.1 s	11.3 s	4.1	1.3e-5
RBM	176 (8)	89.6 s	78.6 s	11 s	7.25	1.1e-5
RBM (SVD)	176 (8)	48.2 s	37 s	11 s	13.5	1.1e-5

Table 12: Analysis result for M =**10**

	$n(n_Q)$	Full time	MOR time	Sweep Time	Speedup	Max. Est.
Original formulation	177981	864 s	—	864 s	—	—
SAPOR	378 (9)	154 s	125 s	29.5 s	5.61	1.2e-5
GM-MOR						
$NF = 10$, $Q_{max} = 10$	378 (9)	427 s	397 s	29.4 s	2.02	6.7e-7
$NF = 20$, $Q_{max} = 5$	378 (9)	475 s	445 s	29.8 s	1.82	1.9e-6
$NF = 40$, $Q_{max} = 2$	403 (10)	611 s	580 s	30.6 s	1.41	3.7e-7
RBM	378 (9)	329 s	299 s	29.6 s	2.63	8e-5
RBM (SVD)	336 (8)	111 s	82 s	29 s	7.78	3.7e-5

5.4. Band 7-13 GHz, 1201 points, tolerance 1e-4, N=177981

Table 13: Analysis result for M =**3**

	$n(n_Q)$	Full time	MOR time	Sweep Time	Speedup	Max. Est.
Original formulation	177981	844 s	—	844 s	—	—
SAPOR	308 (22)	144 s	129 s	15 s	5.86	4.6e-5
GM-MOR						
$NF = 10$, $Q_{max} = 10$	246 (18)	224 s	212 s	12.7 s	3.77	4.2e-7
$NF = 20$, $Q_{max} = 5$	233 (17)	219 s	206 s	12.6 s	3.85	4.7e-6
$NF = 40$, $Q_{max} = 2$	—	—	—	—	—	—
RBM	224 (16)	150 s	138 s	12.4 s	5.63	1.7e-5
RBM (SVD)	210 (15)	92 s	79 s	12.2 s	9.17	6e-5

Table 14: Analysis result for M =**5**

	$n(n_Q)$	Full time	MOR time	Sweep Time	Speedup	Max. Est.
Original formulation	177981	975 s	—	975 s	—	—
SAPOR	462 (21)	250 s	226 s	24.6 s	3.9	3.9e-5
GM-MOR						
$NF = 10$, $Q_{max} = 10$	352 (16)	395 s	375 s	20.6 s	2.47	4.4e-5
$NF = 20$, $Q_{max} = 5$	313 (15)	368 s	348 s	19.8 s	2.65	5.5e-6
$NF = 40$, $Q_{max} = 2$	308 (14)	368 s	349 s	19.8 s	2.65	6.7e-5
RBM	308 (14)	247 s	228 s	19.6 s	3.95	4.4e-5
RBM (SVD)	308 (14)	142 s	122 s	20 s	6.87	9.2e-6

Table 15: Analysis result for M =**10**

	$n(n_Q)$	Full time	MOR time	Sweep Time	Speedup	Max. Est.
Original formulation	177981	1296 s	—	1296 s	—	—
SAPOR	630 (15)	381 s	323 s	57.6 s	3.4	4.7e-5
GM-MOR						
$NF = 10$, $Q_{max} = 10$	558 (14)	999 s	947 s	53 s	1.3	5.1e-5
$NF = 20$, $Q_{max} = 5$	543 (13)	928 s	876 s	52 s	1.4	9.4e-5
$NF = 40$, $Q_{max} = 2$	529 (13)	1072 s	1020 s	52 s	1.2	7.9e-5
RBM	504 (12)	589 s	539 s	50 s	2.2	4.9e-5
RBM (SVD)	546 (13)	324 s	270 s	53 s	4	1.6e-5

5.5. Summary

Big times obtained with GM-MOR algorithm are caused many Gram-Schmidt operations. For the case with total time ca. 1100 s, GS operation take ca. 900 s. Now take into account $M = 5$, $NF = 10$, $Q_{max} = 10$ and both analyzed bandwidths.

Table 16: Analysis result for $M = \mathbf{5}$, $NF = 10$, $Q_{max} = 10$

Band	Problem size	Full time	MOR time	GS time	Estimate Time
9-11	177981	106 s	95 s	78 s	2.5 s
7-13	177981	395 s	375 s	325 s	15.2 s
9-11	15729	24.2 s	13 s	8 s	3.7 s
7-13	15729	59 s	49.1 s	23 s	14 s

As presented in table 16. the biggest time is Gram-Schmidt operation. It is caused by many right hand sides, which are determined by M parameter. Additionally, when moving to the next frequency point, re-orthogonalization of additional vector with current base is necessary, which is also a factor that increases total time.

Results shown that currently fastest algorithm for following case is RBM with SVD algorithm. However, suggestion is to take a look into orthogonalization process.

References

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[2] De La Rubia, Valentin, Ulrich Razafison, and Yvon Maday. *Reliable fast frequency sweep for microwave devices via the reduced-basis method.* IEEE Transactions on Microwave Theory and Techniques 57.12 (2009): 2923-2937.